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Monte Carlo Phase Space with Limited Transverse Momentum

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Techniques are presented for Monte Carlo event generation with limited transverse momentum. The phase space integral is first split into longitudinal and transverse components. General methods are derived for generating transverse momenta according to any distribution given as a product of functions of the transverse momenta of individual particles. The cases of Gaussian, exponential, and power-law distributions are treated. Also discussed is the case where one or more produced particles have high transverse momentum. The longitudinal momenta are generated by decomposing the ensemble of particles via a sequence of two-body decays. A procedure is given for generating according to uniform longitudinal phase space. Also shown are methods for including leading particle behavior or longitudinal clustering. Finally, resonances are incorporated among the particles which may be produced with limited transverse momentum.

1. INTRODUCTION

Large multiplicities and limited transverse momenta are both characteristic of particle reactions at high energy. The large multiplicity n of a reaction requires that the phase space of $3n - 4$ dynamical variables be of rather high dimensionality. The limited transverse momenta mean that only a small fraction of this space will be populated.

The importance of considering both kinematic and dynamic effects in single-particle inclusive reactions has recently been emphasized in an extensive analysis by Taylor *et al.* [1]. The calculation of distributions of one or more kinematic variables involves phase-space integrals of the form

$$J = \int_V F(\rho)\, d^n\rho, \tag{1}$$

where the variable ρ describes a point in the $(3n - 4)$-dimensional space of kinematic

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variables, and V is the region of integration. The integrand F usually takes the form

$$F = |A|^2 G, \tag{2}$$

where A is the relativistically invariant complete matrix element for the reaction. The factor G includes experimental limitations, often represented by step functions, and unintegrated variables, represented by delta functions. It may even involve further integrations over spaces of additional variables, such as in the simulation of electromagnetic or hadronic showers in experimental detectors.

The Monte Carlo method [2] has been recognized as providing the greatest convenience and generality in evaluating integrals of higher dimensionality. The number of function evaluations can be chosen at will, the error formula is known and is easily evaluated, and the validity of the method is not dependent on smoothness conditions of the integrand. Greater efficiency can be obtained if the random points can be distributed in a manner which closely approximates the integrand. The technique of doing this is often given the name "importance sampling." For particle reactions, this means generating events as they occur physically.

The earliest applications of the Monte Carlo method to the simulation of elementary particle reactions were made by Kopylov [3] and by Raubold and Lynch (unpublished) Their techniques produced uniform phase space distributions and relied on a method of ordering the generated sequence of random numbers. Later generalizations were produced by Gillespie [4] and by Friedman [5] which allowed event generation according to a wide variety of distributions. Extensions to multiperipheral models were produced by Byckling *et al.* [6] and by Friedman *et al.* [7].

Methods for direct event generation with damped transverse momentum were first described by Pene and Krzywicki [8] and also by Kittel *et al.* [9]. More recently a third method was derived by Jadach [10]. All three methods generate the transverse and longitudinal momenta separately.

All three methods also use an orthogonal rotation technique to generate the transverse momenta. That of Jadach, however, is much simpler than the others and, as we shall see, much more easily generalized. Kittel *et al.* also use an orthogonal rotation for the longitudinal momenta. They essentially generate directly in the Feynman x variable [11], which results in a weight factor for each event containing in the denominator the product of the energies of the secondaries. For high energies this factor can fluctuate considerably from event to event, thus impairing the efficiency of the method.

Pene and Krzywicki generate the longitudinal momenta via a method which is analogous to that used by Kopylov [3] and by Raubold and Lynch for all three momentum components. Their technique employs an ordering of random numbers, and the resulting weight factor contains poles in the differences of these random numbers. These poles are removed through a rather complicated procedure of transformation of variables. The method is quite efficient, but, owing to the complications in the mathematics, it has not received the appreciation which it deserves.

Jadach generates the longitudinal momenta directly in the rapidity variable

$$y_i = \log[(E_i + p_i^{\mathrm{L}})/(E_i - p_i^{\mathrm{L}})], \tag{3}$$

where E_i and p_i^{L} are the energy and the longitudinal momentum of the ith secondary, respectively. All three methods use a distribution which is given by a product of Gaussian terms in the transverse momentum of the secondaries. The longitudinal momenta in all three cases are given by relativistically invariant phase space. Therefore the methods, as derived, are not suitable for generating according to other distributions, either in transverse or in longitudinal momentum.

A distribution which is Gaussian in the transverse momenta of the secondaries can be generated directly. In fact, if we insist that the distribution be written as the product of factors each dependent on the transverse momentum of a single secondary, then the Gaussian distribution is the only distribution which is invariant under an orthogonal transformation. Therefore the efficiency, which is determined by the variation of the weight factor, is entirely due to the method of generation of longitudinal momenta. The authors have found the method of Pene and Krzywicki to be the most efficient in this sense.

It is desirable to be able to generate events according to a variety of distributions. Physical distributions may differ according to the identity of the particles involved. The consequences of the assumptions of differing matrix elements may also be of theoretical interest. Of interest also is the contribution of different production mechanisms to the totality of a given cross section. Finally, the trigger requirements of a given experiment may restrict the class of interesting events.

Techniques for event generation according to a wide variety of distributions are described here. The separation into transverse and longitudinal components is still made. Transverse momenta may be generated according to distributions which are Gaussian, linear exponential, or power law. One or more particles may be given a minimum transverse momentum to simulate high p^{T} experiments. For the Gaussian case, our method reduces to that of Jadach.

For the longitudinal momenta, our method bears certain similarities to that of Pene and Krzywicki. However, we have derived a procedure whereby the factors which produce the poles in the weight function are included directly and exactly in the generation. Our method is, therefore, both simpler and more flexible than that of Pene and Krzywicki. We can generate according to uniform longitudinal phase space, produce leading particle behavior in one or more secondaries, or produce clustering in longitudinal momentum. We can also allow resonances to be among the peripherally-produced secondaries, including correct calculation of threshold behavior.

The phase-space integral is described quantitatively in Section 2. We review the splitting into transverse and longitudinal factors. Techniques for generating the transverse momenta are developed in Section 3. Those for longitudinal momenta are described in Section 4. The incorporation of resonances into this overall scheme is described in Section 5. Finally, some concluding remarks are made in Section 6.

2. The Phase-Space Integral

The phase-space integral we wish to consider is of the form

$$J = \int_V F(\rho)\, d^n\rho, \tag{4}$$

where the production amplitude and the experimental cuts are contained in the integrand F. The differential phase-space element is given by

$$d^n\rho = \delta^4\left(P_0 - \sum_{i=1}^{n} p_i\right) \prod_{i=1}^{n} \vartheta(p_i^0)\, \delta(p_i^2 - m_i^2)\, d^4p_i\,, \tag{5}$$

where the p_i are the four-momenta of the produced particles and P_0 is the total four-momentum. The volume V consists of the range of momenta kinematically available to the reaction, or some subset thereof defined by the cuts.

We may now decompose the momentum four-vectors into longitudinal and transverse components so that

$$p_i = p_i^{\mathrm{L}} + p_i^{\mathrm{T}} \tag{6}$$

The two transverse momentum components are contained in p^{T}, while the longitudinal momentum and the energy are contained in p^{L}. We choose our metric such that, for a particle on the mass shell, we have

$$p_i^2 = m_i^2. \tag{7}$$

Similarly, we assume that the distribution function F can also be decomposed into what we shall term longitudinal and transverse parts

$$F(\{p^{\mathrm{L}}\}, \{p^{\mathrm{T}}\}) = F_{\mathrm{L}}(\{p^{\mathrm{L}}\}, \{p^{\mathrm{T}}\})\, F_{\mathrm{T}}(\{p^{\mathrm{T}}\}), \tag{8}$$

where $\{p^{\mathrm{L}}\}$ and $\{p^{\mathrm{T}}\}$ are the sets of longitudinal and transverse momentum components of the secondaries. The variation with p^{T} of F_{T} is much stronger than that of F_{L} . The function F_{L} therefore represents any longitudinal structure in the factor F. The authors mentioned above [8–10], who described event generators for an amplitude which was Gaussian in transverse momentum, assumed no longitudinal structure. Therefore, in their case, the longitudinal distribution function F_{L} is just equal to one.

We further assume that the transverse amplitude can be factored into a product of single-particle amplitudes

$$F_{\mathrm{T}}(\{p^{\mathrm{T}}\}) = \prod_{i=1}^{n} f(p_i^{\mathrm{T}}). \tag{9}$$

The differential phase-space element and distribution function may now be decomposed,

$$F\, d\rho = F_{\mathrm{L}}\, d\rho_{\mathrm{L}}\, F_{\mathrm{T}}\, d\rho_{\mathrm{T}}\,, \tag{10}$$

where the individual factors are

$$F_L\, d\rho_L = F_L(\{p^L\}, \{p^T\})\, \delta^2\left(P^L - \sum_{i=1}^{n} p_i{}^L\right)$$

$$\times \prod_{i=1}^{n} \vartheta(p_i{}^0)\, \delta((p_i{}^L)^2 - (m_i{}^L)^2)\, d^2p_i{}^L, \tag{11}$$

$$F_T\, d\rho_T = \delta\left(\sum_{i=1}^{n} u_i\right) \delta\left(\sum_{i=1}^{n} v_i\right) \prod_{i=1}^{n} f(p_i{}^T)\, d^2p_i{}^T. \tag{12}$$

The two transverse momentum components are represented by u and v. The longitudinal masses in Eq. (11) are given by

$$(m_i{}^L)^2 = m_i{}^2 + u_i{}^2 + v_i{}^2. \tag{13}$$

In the following, we first treat the transverse factor, then the longitudinal factor. The longitudinal integral derives some of its p^T dependence through the presence of the longitudinal masses in the mass shell delta functions. Therefore computationally it is necessary to perform the integrals in the same order. The transverse momenta are first obtained, giving the longitudinal masses. Then the longitudinal component is evaluated, yielding a complete set of four-momenta.

In the case of both factors we transform variables to those whose range is from zero to one. The events can then be generated using sets of uniformly distributed random numbers for these variables. In each case, we will be left with a weighting function. It will be given by the product of the distribution function at the generated point and the Jacobian of the variable transformation. The total weight for the event is given by the product of the transverse and longitudinal parts, so that

$$W = W^T W^L. \tag{14}$$

3. The Transverse Components

A. *General Considerations*

(i) *Preliminary Remarks*

The transverse momentum components may be generated according to any factorizable, integrable distribution. In fact, if it is possible to generate the single-particle inclusive transverse momentum distribution, it is possible to generate a complete set of four-momenta, including momentum conservation.

We begin by ignoring energy-momentum conservation and generate a complete set of single-particle transverse momenta. These preliminary transverse momenta $q_i{}^T$ are then transformed to a new set $p_i{}^T$ which satisfy momentum conservation. A weight

factor is calculated which contains both the effect of the imposition of momentum conservation and the Jacobian of the transformation to the space of random numbers. Possibilities for determining the most efficient method of generation are explored. The method is then applied to distributions which are Gaussian, exponential, or power law in transverse momentum. Finally we consider the case where one or more produced particles are constrained to have a transverse momentum above a specified minimum value.

(ii) *Mathematical Procedure*

The preliminary transverse momenta q_i^{T} are distributed according to

$$f(q_i^{\mathrm{T}})\, q_i^{\mathrm{T}}\, dq_i^{\mathrm{T}}.$$

The most straightforward way of generating such momenta is to integrate directly the product $f(q_i^{\mathrm{T}})\, q_i^{\mathrm{T}}$ and invert the result. In terms of a random number r_1 derived from a uniform distribution from 0 to 1, the magnitude of the transverse momentum is then given by

$$\int_0^{q_i^{\mathrm{T}}} f(\zeta)\, \zeta\, d\zeta = r_1 I, \tag{15}$$

where the factor I represents the integral from 0 to ∞, so that

$$I = \int_0^{\infty} f(\zeta)\, \zeta\, d\zeta. \tag{16}$$

The equation resulting from the integration is solved for the magnitude of q^{T}. The angle φ of the q^{T} vector in the plane of the two q^{T} components is given by a second uniformly distributed random number r_2,

$$\varphi = 2\pi r_2 . \tag{17}$$

The complete set of transverse momentum components then requires $2n$ random numbers, two for the transverse momentum of each particle.

Not all interesting transverse momentum distributions can be treated in such a manner. In some cases the integration represented in Eq. (15) results in a transcendental equation which cannot be solved analytically. In specific cases there may be techniques which yield transverse momenta more efficiently than an iterative numerical solution of Eq. (15). Techniques for individual cases are discussed in detail below.

The components of the generated transverse momenta will be denoted by u' and v'. We now follow the procedure of Jadach [10] and derive new quantities u and v so that

$$\begin{aligned} u' &= u + \xi a, \\ v' &= v + \eta a, \end{aligned} \tag{18}$$

where, for example,

$$u = (u_1, u_2, \ldots, u_n),$$
$$a = (1/n^{1/2})(1, 1, \ldots, 1), \tag{19}$$

and

$$u \cdot a = v \cdot a = 0. \tag{20}$$

The individual transverse momenta of the secondaries are now given by the u and v variables. The untransformed momenta are distributed according to the function $f(q^{\mathrm{T}})$, while the transformed momenta satisfy the delta function constraints. Therefore we must weigh each event by the factor

$$W_1 = \prod_{i=1}^{n} f(p_i^{\mathrm{T}}) \Big/ \prod_{i=1}^{n} f(q_i^{\mathrm{T}}), \tag{21}$$

where the p_i^{T} are the final values of p^{T}, and the q_i^{T} are those before the transformation of variables.

The weight function is equal to one when ξ and η are zero, which is the most likely value of these two variables. However, the transformation of variables represented in Eqs. (18) transforms some of the transverse damping to the discarded coordinates ξ and η. This damping is divided out in the denominator but is not reimposed in the numerator of the expression for W_1. Therefore, on the average W_1 will be an increasing function of ξ and η. Chen and Peierls [12] have shown that the weight factor may be multiplied by arbitrary normalized functions of ξ and η. Because of the cylindrical symmetry of the problem, we choose to use a function of $(\xi^2 + \eta^2)^{1/2}$ and define a new weight

$$W_0 = g[(\xi^2 + \eta^2)^{1/2}] \left[\prod_{i=1}^{n} f(p_i^{\mathrm{T}}) \Big/ \prod_{i=1}^{n} f(q_i^{\mathrm{T}}) \right]. \tag{22}$$

The discarded coordinates ξ and η are each proportional to the sum of n random variables generated according to n independent identical distributions $f(p_i^{\mathrm{T}})$. Therefore the central limit theorem suggests that the form of the function $g(\zeta)$ should be Gaussian,

$$g(\zeta) = (1/\pi\sigma^2)\, e^{-\zeta^2/\sigma^2}. \tag{23}$$

The quantity σ^2 is the normalized second moment of the distribution function $p^{\mathrm{T}} f(p^{\mathrm{T}})$, or

$$\sigma^2 = \int_0^\infty p^{\mathrm{T}} f(p^{\mathrm{T}})\, dp^{\mathrm{T}} \Big/ \int_0^\infty p^{\mathrm{T}} f(p^{\mathrm{T}})\, dp^{\mathrm{T}}. \tag{24}$$

Chen and Peierls have also shown that the greatest event generation efficiency will occur when in fact

$$g(\zeta) \propto [\overline{W}_1(\zeta)/\overline{W_1^2}(\zeta)], \tag{25}$$

where the statistical means indicated are taken with ζ fixed. Since $W_1(0)$ is constant, it is expected that the optimal $g(\zeta)$ should then drop off with increasing ζ more rapidly than indicated by Eq. (23). However, in practice, the optimal average generation efficiency is not the only relevant consideration. It is also important that the efficiency be robust. This means that we must ensure that the efficiency estimated from any reasonable statistical sample will be close to optimal. For most forms of the distribution function f, the weight function W_1, as defined in Eq. (21), is not strictly a function of its argument ζ. In general, for a given value of ζ, the value of W_1 will fluctuate over a range. Since W_1 is constant when ζ is equal to zero, this range of fluctuations will, in general, expand as the value of ζ increases. An anomaloulsy high value of W_1 in a statistical sample will produce a very low efficiency. Therefore it is best to ensure that such values do not occur.

The size of the maximum fluctuations has been found to be approximately a function of the magnitude of the sum of the initially generated two-momenta. This corresponds to the quantity $n^{1/2}\zeta$. Therefore, in certain specific cases, as described below, we have introduced a multiplicity-dependent cutoff in the function g. In these cases we replace Eq. (23) by

$$g(\zeta) = [e^{-\zeta^2/\sigma^2}/\pi\sigma^2(1 - e^{-Z^2/\sigma^2 n})]\, \vartheta(Z - \zeta n^{1/2}), \tag{26}$$

where Z is the value of the cutoff parameter described above. The function ϑ is the usual step function which is equal to one above zero and zero below.

To determine the u portion of the differential phase-space element, we can formally express the u variables in an orthonormal system where the first $n - 1$ coordinates $\{s_i\}$ span the subspace allowed by the δ function. The remaining coordinate λ is measured in the direction of a. We then have

$$\begin{aligned} d\rho_u &= \delta\left(\sum_{i=1}^{n} u_i\right) \prod_{i=1}^{n} du_i \\ &= \delta(n^{1/2}\lambda)\, d\lambda \prod_{i=1}^{n-1} ds_i\,. \end{aligned} \tag{27}$$

Integrating over λ, we arrive at

$$d\rho_u = \frac{1}{n^{1/2}} \prod_{i=1}^{n-1} ds_i\,. \tag{28}$$

The total weight for transverse momentum generation will contain a factor of $(2\pi I)^n$, owing to the Jacobian of the transformation to the space of random numbers, and a factor of $1/n$ from the transformation of variables to satisfy momentum conservation. Then the transverse contribution to the weight function for each event will become

$$W^{\mathrm{T}} = \frac{1}{n}(2\pi I)^n\, g[(\xi^2 + \eta^2)^{1/2}] \left[\prod_{i=1}^{n} f(p_i{}^{\mathrm{T}}) \Big/ \prod_{i=1}^{n} f(q_i{}^{\mathrm{T}})\right]. \tag{29}$$

Since the transverse momenta are all generated from zero to infinity, it is possible for the energy used in generating the transverse momenta to exceed the total energy available to the reaction. If this happens, we simply discard the set of generated momenta and try again. In principle, then, the weight functions should be derived from integrals taken only over the kinematically accessible region. However, this is analytically very difficult, if not impossible. At high energies, the approximations used will be very good. A rejected set of transverse momenta will occur only extremely rarely, and the error from the integrals will be much smaller than the statistical error from any Monte Carlo sample.

B. *Specific Transverse Momentum Distributions*

(i) *The Gaussian Distribution*

Gaussian damping in transverse momentum is the case most extensively treated in the literature [8–10]. In this case we have

$$f(p^{\mathrm{T}}) = e^{-(p^{\mathrm{T}})^2/R^2}. \tag{30}$$

If we choose the function $g(\zeta)$ according to Eq. (23), our method reduces to that of Jadach. The weight function is constant and is given by

$$W^{\mathrm{T}} = (1/n)(2\pi I)^{n-1}, \tag{31}$$

where

$$I = \tfrac{1}{2}R^2. \tag{32}$$

We can perform the integral in Eq. (15) exactly, obtaining

$$\tfrac{1}{2}r_1R^2 = \tfrac{1}{2}R^2(1 - e^{-(p^{\mathrm{T}})^2/R^2}) \tag{33}$$

or, if we solve for p^{T} and redefine r_1,

$$p^{\mathrm{T}} = R(-\log r_1)^{1/2}. \tag{34}$$

(ii) *Linear Exponential in Transverse Momentum*

The distribution considered here takes the form

$$f(p^{\mathrm{T}}) = e^{-p^{\mathrm{T}}/a}. \tag{35}$$

We have chosen a form such that the constant a, as in the previous case, will have the dimensions of momentum. The integral in Eq. (15) becomes

$$\int_0^{p^{\mathrm{T}}} e^{-\zeta/a}\zeta \, d\zeta = a^2\left[1 - \left(1 + \frac{p^{\mathrm{T}}}{a}\right)e^{-p^{\mathrm{T}}/a}\right], \tag{36}$$

which gives a value for the definite integral from 0 to ∞ of

$$I = a^2. \tag{37}$$

However, since Eq. (36) is transcendental, it is not analytically solvable for p^T. We may circumvent this difficulty by a procedure of Everett and Cashwell [13]. Here we use three random numbers r_1, r_2, and r_3, each with uniform distribution from 0 to 1, to generate the two transverse momentum components of a single particle. The magnitude of the transverse momentum is given by

$$p^T = -a \log(r_1 r_2), \tag{38}$$

while the angle in the x–y plane is given by

$$\varphi = 2\pi r_3 . \tag{39}$$

In this case the weight function W_1, given in Eq. (21), is not strictly a function of $(\xi^2 + \eta^2)^{1/2}$, and we use the cutoff form of the function g as defined in Eq. (26). The quantity σ^2 is given by

$$\sigma^2 = 6a^2. \tag{40}$$

We find that a value of $8a$ for the cutoff parameter Z will ensure robustness of the efficiency, without significantly compromising its average value. The weight function now becomes

$$W^T = \frac{1}{n} (2\pi I)^{n-1} \frac{e^{-(\xi^2+\eta^2)/6a^2}}{3(1 - e^{-64/6n})} \left[\prod_{i=1}^{n} e^{-p_i T/a} \Big/ \prod_{i=1}^{n} e^{-q_i T/a} \right] \vartheta[Z - n^{1/2}(\xi^2 + \eta^2)^{1/2}]. \tag{41}$$

(iii) *Power Law in Transverse Momentum*

A strict power law in p^T would have a singularity at zero, thus representing a physically meaningless cross section. We therefore use the amplitude discussed in Ref. [1], where

$$f(p^T) = m^{2\alpha}/[(p^T)^2 + m^2]^{\alpha}. \tag{42}$$

The constant m need have no relation to the mass of the particle but is chosen to give the desired distribution. For the function f to be integrable from 0 to ∞, we must require that $\alpha > 1$. The normalization is chosen so that, as in the previous two cases, $f(0)$ will be equal to 1. In this case we can perform the integral in Eq. (15) and solve the resulting equation directly for p^T, obtaining

$$p^T = m \left[\frac{1}{r_1^{1/(\alpha-1)}} - 1 \right]^{1/2}. \tag{43}$$

The integral I, as defined in Eq. (16), is given by

$$I = m^2/2(\alpha - 1). \tag{44}$$

Here, once again, we use the cutoff form of the normalized function g. The value of σ^2 as derived from Eq. (24) is now given by

$$\sigma^2 = m^2/(\alpha - 2). \tag{45}$$

A value of 6σ for the cutoff parameter Z has been found to minimize the effect of fluctuations in the quantity W_1. The total transverse weight for each event is therefore given by Eq. (29), with no simplification in form.

C. *High Transverse Momentum Particles*

(i) *General Procedures*

The production of one or more secondaries with large p^{T} is often of interest. It may be the trigger for an experiment [14], or it may be of theoretical interest as isolating certain production mechanisms [15]. We discuss here the case where a single particle is required to have a transverse momentum greater than some specified minimum $p^{\mathrm{T}}_{\min}$.

We choose particle number 1 as the secondary which is required to have high transverse momentum. Its transverse momentum $p_1{}^{\mathrm{T}}$ is obtained from the distribution f with the constraint that it must exceed $p^{\mathrm{T}}_{\min}$. In terms of a random number with a uniform distribution from 0 to 1, $p_1{}^{\mathrm{T}}$ is given by

$$\int_{p^{\mathrm{T}}_{\min}}^{p_1{}^{\mathrm{T}}} f(\zeta)\, d\zeta = r_1 I_1\,, \tag{46}$$

where now

$$I_1 = \int_{p^{\mathrm{T}}_{\min}}^{\infty} f(\zeta)\, \zeta\, d\zeta. \tag{47}$$

We now generate and transform the transverse momenta of the remaining $n - 1$ particles in a manner similar to that of the previous case. The two components u' and v' of the transverse momentum are generated from the distribution function f. They are then transformed to new momenta u and v via

$$\begin{aligned} u' &= u + \left[\xi + \frac{1}{(n-1)^{1/2}}\, p_{1x}\right] a, \\ v' &= v + \left[\eta + \frac{1}{(n-1)^{1/2}}\, p_{1y}\right] a, \end{aligned} \tag{48}$$

where now

$$a - \frac{1}{(n-1)^{1/2}}\,(0, 1, 1, \ldots, 1). \tag{49}$$

The values of ξ and η are chosen so that

$$\begin{aligned} \xi &= u' \cdot a, \\ \eta &= v' \cdot a. \end{aligned} \tag{50}$$

Proceeding as before, we obtain, for the total transverse weight for the event,

$$W^{\mathrm{T}} = \frac{1}{n-1} (2\pi I)^{n-1} (2\pi I_1)\, g[(\xi^2 + \eta^2)^{1/2}] \left[\prod_{i=2}^{n} f(p_i{}^{\mathrm{T}}) \Big/ \prod_{i=2}^{n} f(q_i{}^{\mathrm{T}}) \right]. \tag{51}$$

Once again the q_i are the untransformed transverse momenta initially generated, while the p_i are the transformed momenta satisfying momentum conservation. The products go from 2 to n, since the momentum of particle 1 is unaffected by the transformation from the q's to the p's.

(ii) *The Gaussian Distribution*

For the Gaussian distribution the integral in Eq. (46) may be performed directly and solved for $p_1{}^{\mathrm{T}}$. In terms of r_1, we have

$$p_1{}^{\mathrm{T}} = [(p_{\min}^{\mathrm{T}})^2 - R^2 \log r_1]^{1/2}. \tag{52}$$

The total transverse weight takes the form

$$W^{\mathrm{T}} = [1/(n-1)](2\pi I)^{n-2}(2\pi I_1), \tag{53}$$

where

$$I_1 = \tfrac{1}{2} R^2 e^{-(p_{\min}^{\mathrm{T}})^2/R^2}. \tag{54}$$

(iii) *Linear Exponential*

For the linear exponential, Eq. (46) may be integrated, but it cannot be solved analytically for $p_1{}^{\mathrm{T}}$. In the case where $p_{\min}^{\mathrm{T}}$ was equal to zero, we obtained the required p^{T} from a set of two random numbers according to Eq. (38). The set of two random numbers with uniform distribution from 0 to 1 may be regarded as a point in a unit square with uniform probability distribution.

Now we wish to generate according to the same distribution f, but subject to the constraint $p_1{}^{\mathrm{T}} > p_{\min}^{\mathrm{T}}$. This means that we wish once again to generate a point with uniform distribution in a unit square, but with a segment removed. That segment is the region where

$$r_1 r_2 > e^{-p_{\min}^{\mathrm{T}}/a}. \tag{55}$$

The region from which points are to be taken is shown as the shaded area in Fig. 1.

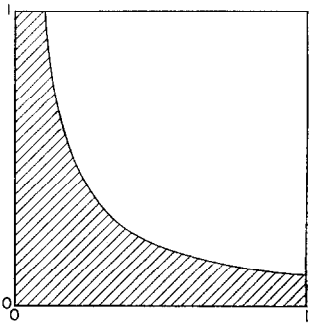

FIG. 1. Domain of the pair of random numbers used to generate exponentially distributed transverse momenta with nonzero minimum. The two random numbers r_1 and r_2 are given uniform distribution in the cross-hatched area.

We define a quantity P via

$$P = e^{-p_{\min}^{T}/a} \tag{56}$$

Then the area A of the shaded region is given by

$$A = P - P \log P. \tag{57}$$

We define two new random numbers ρ_1 and ρ_2, generated uniformly between 0 and 1. If $\rho_1 A \leqslant P$, then we obtain r_1 and r_2 as

$$\begin{aligned} r_1 &= \rho_1 A, \\ r_2 &= \rho_2 . \end{aligned} \tag{58}$$

If $\rho_1 A > P$, then we have

$$\begin{aligned} r_1 &= P e^{(1/P)\rho_1 A - 1}, \\ r_2 &= \rho_2 P / r_1 . \end{aligned} \tag{59}$$

Then we have, as before,

$$p_1{}^{T} = -a \log(r_1 r_2). \tag{60}$$

The remaining momenta are generated as described above. Once again we choose the function g by Eq. (23) and arrive at a final weight of

$$W^{T} = \frac{1}{n-1} (2\pi I)^{n-2} (2\pi I_1) \frac{e^{-(\xi^2+\eta^2)/6a^2}}{3(1 - e^{-64/6n})} \left[\prod_{i=2}^{n} e^{-p_i{}^{T}/a} \Big/ \prod_{i=2}^{n} e^{-q_i{}^{T}/a} \right] \times [Z - n^{1/2}(\xi^2 + \eta^2)^{1/2}], \tag{61}$$

with

$$I_1 = a^2 e^{-p_{\min}^{T}/a} \left(1 + \frac{p_{\min}^{T}}{a}\right). \tag{62}$$

(iv) *Power Law*

The power-law distribution described earlier can be generated directly. In terms of $p_{\min}^{T}$ and r_1, the momentum $p_1{}^{T}$ is given as

$$p_1{}^{T} = \left[\frac{(p_{\min}^{T})^2 + m^2}{r_1^{1/(\alpha-1)}} - m^2\right]^{1/2}. \tag{63}$$

We choose the normalizing function g as in the case with $p_{\min}^{T}$ equal to 0. Then the transverse weight function is given just as in Eq. (51) with

$$I_1 = \frac{m^{2\alpha}}{2(\alpha - 1)[(p_{\min}^{T})^2 + m^2]^{\alpha-1}}. \tag{64}$$

4. The Longitudinal Components

A. *General Considerations*

We begin by decomposing the n-particle ensemble via a succession of quasi-two-body decays into n single-particle states. The manner of splitting may be chosen either to represent some physical mechanism, or for mathematical convenience. The variables describing this decomposition are then the invariant masses of the intermediate multiparticle states.

These invariant masses are then related to another set of variables with range from zero to one. The distribution of these variables as required by phase-space kinematics is derived below.

The generated distribution of random numbers used for these variables will depend on both phase space and the longitudinal distribution function F. Once the invariant masses are known, the center-of-mass momenta at a given vertex can be calculated. These momenta may then be Lorentz transformed to the laboratory frame, which gives a complete set of particle four-momenta.

For uniform longitudinal phase space the method of decomposition and the distribution of the associated random numbers giving maximum efficiency are derived. We compare the efficiency of our method with that of other authors and find that we have achieved a considerable improvement at high multiplicity.

Leading particle distributions may be imposed by breaking single particles away from the remainder and giving them any distribution desired. The remaining ensemble may then be given uniform longitudinal distribution. We derive methods for generating the leading particle momenta and calculate the contribution to the longitudinal weight function.

Finally, any longitudinal structure can be incorporated into our scheme. We give an example where the secondary particles are produced in two clusters, with the mass

of each exponentially damped. These masses are then generated in a manner which includes both the damping and the phase-space factors. The final longitudinal weight is a product of factors which vary slowly over the available kinematic range.

B. *Phase-Space Factorization*

The n-body longitudinal phase-space element $d\rho_n$ (with the subscript L temporarily dropped) may be factored in a manner similar to that commonly used for n-body uniform phase space [16]. The ensemble of n particles can be decomposed into two ensembles of k and $n - k$ particles as follows.

$$d\rho_n = (1/2q_k W_n)\, d\mu_k^2\, d\mu_{n-k}^2\, \vartheta(q_k^2)\, d\rho_k\, d\rho_{n-k}\,. \tag{65}$$

The quantities μ_k and μ_{n-k} represent the invariant masses of the ensembles of k and $n - k$ particles, respectively. The center-of-mass momentum of the ensembles is q_k, while the total energy of the system of n particles is W_n.

We call attention, at this point, to a discrepancy of a factor of 2 from what would be obtained by straight transformation of variables. This is a "solid angle factor" which represents the surface measure of a one-dimensional sphere (i.e., two points). Physically it is caused by the existence of a twofold choice of the direction of motion of the factored parts of the original ensemble.

We can continue to decompose the remaining multiparticle ensembles until the phase-space element is reduced to single-particle states. The decomposition can then be represented by a tree, as shown in Fig. 2. The splitting can be done in any manner desired. The only requirement is that momentum and energy be conserved at every stage. Reasons for preferring one topology to another will be discussed in conjunction with specific applications below.

The phase space element $d\rho_n$ now reduces to

$$\begin{aligned} d\rho_n &= \delta(\mu_n^2 - E_0^2)\, \vartheta(\text{connected}) \prod_{i=2}^{n} \frac{1}{2\mu_i q_i}\, d\mu_i^2\, \vartheta(q_i^2) \\ &= \delta(\mu_n^2 - E_0^2)\, \vartheta(\text{connected}) \prod_{i=2}^{n} \frac{1}{q_i}\, d\mu_i\, \vartheta(q_i^2). \end{aligned} \tag{66}$$

The factor ϑ (connected) merely indicates that the decomposition is to be done according to the tree structure specified. A tree representing the decomposition of an n-particle state will have $n - 1$ nodes, and therefore $n - 2$ invariant masses to generate. With each node we associate the mass of the immediately upstream branch. The mass upstream of the primary node is just the total energy E_0 of the system, which gives rise to the delta function in Eq. (66).

For Monte Carlo event generation we must finally transform the invariant mass variables to a set of $n - 2$ variables each with a range from zero to one. These latter

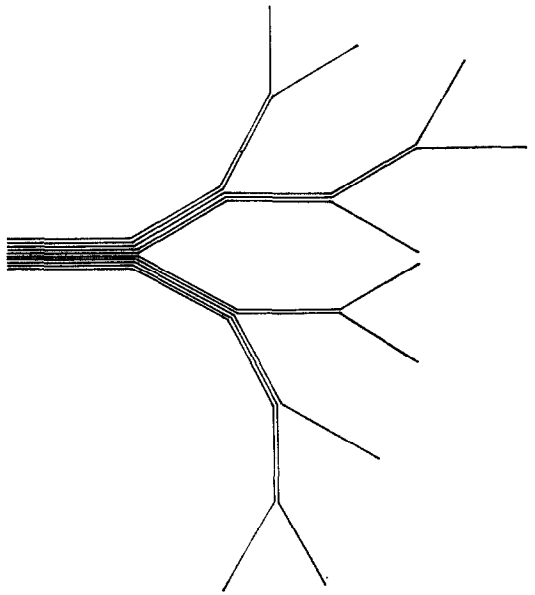

FIG. 2. Tree used to generate the longitudinal momenta of the secondary particles. The initial n-body state is reduced to a set of single-body states via a sequence of quasi-two-body decays.

variables are the set of uniformly distributed random numbers used in event generation. The Jacobian of the transformation to these variables is contained in the longitudinal weighting factor.

The mass of each branch is generated from the range of masses kinematically available. To illustrate, we isolate in Fig. 3 a segment of the tree shown in Fig. 2.

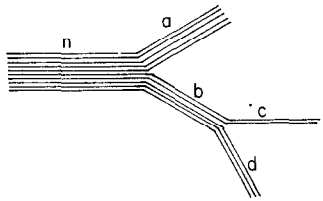

FIG. 3. A segment of the tree shown in Fig. 2. Two nodes are shown and the branches are labeled in the order in which the masses are generated. In the text it is shown how the range of masses available to a given branch depends on the masses of the preceding branches.

As before we denote the invariant mass of each branch as μ_i and the sum of the longitudinal masses of the particles comprising the branch as S_i. In the figure we generate the masses of the branches in alphabetical order. We also associate with each branch a random variable τ_i which may range from 0 to 1, but whose distribution is as yet undetermined.

We may now define the invariant masses μ in terms of the variables τ by

$$\begin{aligned}
\mu_a &= (\mu_n - S_a - S_b)\,\tau_a + S_a\,,\\
\mu_b &= (\mu_n - \mu_a - S_b)\,\tau_b + S_b\,,\\
\mu_c &= (\mu_b - S_c - S_d)\,\tau_c + S_c\,,\\
\mu_d &= (\mu_b - \mu_c - S_d)\,\tau_d + S_d\,,
\end{aligned} \tag{67}$$

so that

$$\begin{aligned}
d\mu_a &= (\mu_n - S_a - S_b)\,d\tau_a\,,\\
d\mu_b &= (\mu_n - \mu_a - S_b)\,d\tau_b\,,\\
d\mu_c &= (\mu_b - S_c - S_d)\,d\tau_c\,,\\
d\mu_d &= (\mu_b - \mu_c - S_d)\,d\tau_d\,.
\end{aligned} \tag{68}$$

The quantity in parentheses in each equation, which multiplies the τ variable, is the factor which contributes to the Jacobian of the transformation from the μ variables to the τ variables. It is the measure of the range kinematically available to the mass being generated. Except in the case of the first mass, this range is variable and would contribute to variations in the weight function. Such variations would then decrease the overall event generation efficiency.

However, in each equation containing a μ variable on the right-hand side, we may substitute the expression for that μ variable from the equations above it. We then derive

$$\begin{aligned}
d\mu_a &= (\mu_n - S_n)\,d\tau_a\,,\\
d\mu_b &= (\mu_n - S_n)(1 - \tau_a)\,d\tau_b\,,\\
d\mu_c &= (\mu_n - S_n)(1 - \tau_a)\,\tau_b\,d\tau_c\,,\\
d\mu_d &= (\mu_n - S_n)(1 - \tau_a)\,\tau_b(1 - \tau_c)\,d\tau_d\,.
\end{aligned} \tag{69}$$

The center-of-mass momenta q_i in Eq. (66) occur in the denominator of the weight factor. These momenta can have the value zero when the associated masses are at the extreme of their kinematic range. We therefore have a set of poles in the weight function which would then, at the very least, damage the efficiency of the event generation.

Examining in detail, for example, the center-of-mass momentum at the node downstream of branch b in Fig. 3, we have

$$q_b = \frac{[(\mu_b + \mu_c + \mu_d)(\mu_b - \mu_c + \mu_d)(\mu_b + \mu_c - \mu_d)(\mu_b - \mu_c - \mu_d)]^{1/2}}{2\mu_b}\,. \tag{70}$$

Defining a new quantity R according to

$$R_b = [(\mu_b + \mu_c + \mu_d)(\mu_b - \mu_c + \mu_d)(\mu_b + \mu_c - \mu_d)]^{1/2}, \tag{71}$$

we then have

$$q_b = (R_b/2\mu_b)(\mu_b - \mu_c - \mu_d)^{1/2}, \tag{72}$$

where now the second factor is the only one which can equal zero, and it therefore yields the pole in the weight function. However, we also have

$$\mu_b - \mu_c - \mu_d = (\mu_n - S_n)(1 - \tau_a)\,\tau_b(1 - \tau_c)(1 - \tau_d). \tag{73}$$

We may now rewrite Eq. (66) as

$$d\rho_n = \frac{1}{2E_0}\,\vartheta(\text{connected})(E_0 - S_n)^{(n-3)/2} \prod_{i=2}^{n} \frac{2\mu_i}{R_i} \prod_{i=1}^{n-1} \tau_i^{j_i}(1 - \tau_i)^{j_i'}\, d\tau_i\,. \tag{74}$$

The exponent j_i' receives contributions as follows:

(a) 1 for each branch whose mass increases with τ_i,

(b) $-\frac{1}{2}$ for each node whose center-of-mass momentum increases with τ_i.

The exponent j_i receives contributions from

(a) 1 for each branch whose mass decreases with τ_i,

(b) $-\frac{1}{2}$ for each node whose center-of-mass momentum decreases with τ_i.

We now consider a branch of k particles which is decomposed into branches of k_1 and k_2 particles, as shown in Fig. 4. Downstream of branch a, one would find $k_1 - 1$ nodes and $k_1 - 2$ multiparticle branches. Downstream of branch b, one would find $k_2 - 1$ nodes and $k_2 - 2$ multiparticle branches. If we generate μ_a first, all momenta and masses emanating from either branch depend on τ_a. If we generate μ_b second, only those momenta and masses emanating from branch b, plus the center-of-mass momentum at the node shown, depend on τ_b. Therefore, in Eq. (74) the factors associated with τ_a are $\tau_a^{(k_1-3)/2}(1 - \tau_a)^{(k_2-2)/2}$, and those associated with τ_b are $\tau_b^{(k_2-3)/2}(1 - \tau_b)^{-1/2}$.

Below we describe methods of generation of the invariant masses of the branches which are appropriate to different longitudinal distribution functions. Once the invariant masses of both branches leading from a given node are known, the center-of-mass momentum of the branches at that node may be calculated through Eq. (70).

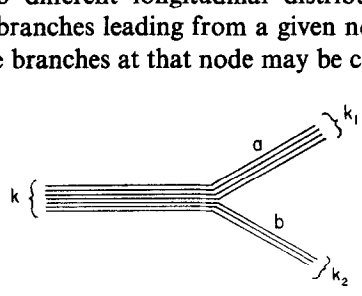

FIG. 4. A single multiparticle branch is split into two multiparticle branches. The mass distribution of the first branch depends on the number of particles in both branches. The mass distribution of the second branch depends on the number of particles in that branch alone.

The four-momenta of these branches in any other reference system may then be obtained via a Lorentz transformation.

In particular, one may obtain the four-momenta of all branches in the center-of-mass system of the entire reaction. In this case, all Lorentz transformations are longitudinal and involve only longitudinal momentum components. The transformations may be derived iteratively. If the longitudinal momentum and energy of a branch in the center of mass of the immediately upstream branch are q and ϵ, respectively, the corresponding components p and E in the total center of mass are given by

$$\begin{aligned} p &= \gamma q + \eta\epsilon, \\ E &= \gamma\epsilon + \eta q. \end{aligned} \tag{75}$$

The factors γ and η are given in terms of the momentum p', energy E', and invariant mass M of the upstream branch by

$$\begin{aligned} \gamma &= E'/M, \\ \eta &= p'/M. \end{aligned} \tag{76}$$

Starting with the entire ensemble of n particles, where γ is 1 and η is 0, we can proceed down the branches of the tree. At each node we generate whatever invariant masses are as yet undetermined and Lorentz transform the resulting four-momenta. The terminal branches will be the final-state particles whose four-momenta will then have been generated.

C. *Uniform Longitudinal Phase Space*

The first case of interest is clearly that of uniform longitudinal phase space where the longitudinal distribution function F_L, defined in Eq. (8) is unity. Therefore we wish to generate the numbers τ_i according to the distributions derived in the previous section. When an ensemble of k particles is split into two ensembles of k_1 and k_2 particles, we wish to define τ_a and τ_b in terms of two random numbers r_1 and r_2, uniformly distributed from zero to one, so that

$$\begin{aligned} I_1 \frac{dr_1}{d\tau_a} &= \tau_a^{(k_1-3)/2}(1 - \tau_a)^{(k_2-2)/2}, \\ I_2 \frac{dr_2}{d\tau_b} &= \tau_b^{(k_2-3)/2}(1 - \tau_b)^{-1/2}. \end{aligned} \tag{77}$$

The two integrals I_1 and I_2 are defined so that both the τ variables and the r variables vary from zero to one. They are given by

$$\begin{aligned} I_1 &= \int_0^1 \tau_a^{(k-3)/2}(1 - \tau_a)^{(k_2-2)/2}\, d\tau_a\,, \\ I_2 &= \int_0^1 \tau_b^{(k_2-3)/2}(1 - \tau_b)^{-1/2}\, d\tau_b\,. \end{aligned} \tag{78}$$

The solution for the τ variables in terms of the r variables may be performed analytically if we choose k_2 equal to 2 and k_1 equal to $k - 2$. Then we have

$$\begin{aligned} \tau_a &= r_1^{2/(k_1-1)}, \\ \tau_b &= \tfrac{1}{2}[1 + \sin \pi(r_2 - \tfrac{1}{2})], \end{aligned} \tag{79}$$

and

$$\begin{aligned} I_1 &= 2/(k_1 - 1), \\ I_2 &= \pi. \end{aligned} \tag{80}$$

Therefore we proceed by splitting off particles two at a time, as shown in Fig. 5. Each such split contributes a factor of $I_1 I_2$ to the longitudinal weight, except when a set of three particles is split into a set of two plus a single particle. In that case $k_1 = 1$, and the mass of one branch is already given. Therefore only a factor of I_2 is contributed to the longitudinal weight. We now have

$$W^L = \frac{1}{2E_0} (E_0 - S_n)^{(n-3)/2} \left[\prod_{i=2}^{n} \frac{2\mu_i}{R_i} \right] I_w , \tag{81}$$

where I_w contains the factors of I_1 and I_2 as described above.

The twofold choice of direction of the split may be made with the aid of another random number. The center-of-mass momenta are obtained from the generated masses, and the laboratory momenta and energies are finally obtained by Lorentz transformation.

We compare the efficiency of our method with that of Pene and Krzywicki [8] for the reaction $p + p \to p + p + (n - 2)\pi$ at 200 GeV/c. We may evaluate the efficiency using the expression given by Eq. (30). If we generate the transverse momenta with a Gaussian distribution, the transverse component of the total weight will be constant. The efficiency obtained will then be entirely a reflection of the method of generation of longitudinal momenta.

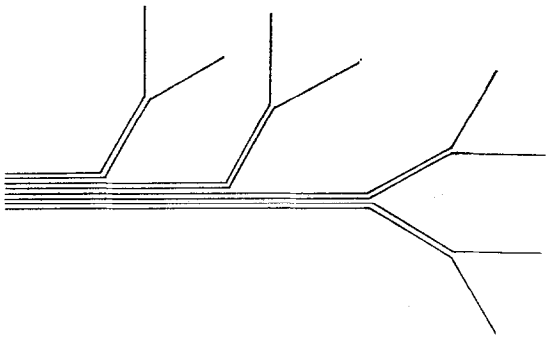

FIG. 5. The tree used to generate longitudinal momenta with uniform phase space distribution. The particles are split off from the remainder two at a time.

We find that for low multiplicities the efficiencies of our method and that of Pene and Krzywicki are comparable. At a multiplicity of 4, the efficiency of the method of Pene and Krzywicki is about 90%, while that of our method is about 80%. However, as the multiplicity is increased, the efficiency of the method of Pene and Krzywicki decreases, reaching a value of approximately 20% for a multiplicity of 24. The efficiency of our method changes very little and is still approximately 60% for a multiplicity of 24. A plot of the efficiencies of the two methods versus multiplicity may be seen in Fig. 6. It is clear that the exact treatment of the τ factors in our method offers a considerable improvement in efficiency at high multiplicity over the pole-

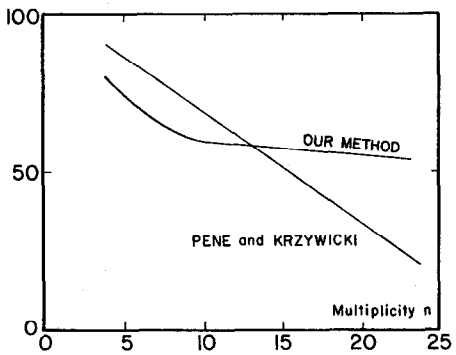

FIG. 6. The efficiencies for generation of longitudinal momenta are shown as a function of multiplicity. Given are the results for our method and for that of Pene and Krzywicki.

D. *Leading Particle Behavior*

Experimentally [17] the longitudinal momentum distribution of protons in the products of proton–proton collisions is quite different from that given by uniform longitudinal phase space. It is desirable to be able to generate events where the distribution of the longitudinal momentum of the final state protons is much more uniform in Feynman x. This can be done by choosing the longitudinal distribution function in Eq. (10) as follows.

$$F_{\mathrm{L}} = E_1 E_n \,, \tag{82}$$

where E_1 and E_n are the center-of-mass energies of the first and nth secondaries. If we now integrate out the individual particle energies we derive

$$F_{\mathrm{L}}\, d\rho_{\mathrm{L}} = \delta^2\left(P^{\mathrm{L}} - \sum_{i=1}^{n} p_i{}^{\mathrm{L}}\right) E_1 E_n \prod_{i=1}^{n} \frac{dp_i{}^{\mathrm{L}}}{2E_i} \,, \tag{83}$$

which may be rewritten as

$$F_L\, d\rho_L = \frac{1}{4}\,\delta^2\left(P^L - p_1{}^L - p_n{}^L - \sum_{i=2}^{n-1} p_i{}^L\right) dp_1{}^L\, dp_n{}^L \prod_{i=2}^{n-1} \frac{dp_i{}^L}{2E_i}\,. \tag{84}$$

The term on the right side is just $\frac{1}{4}dp_1{}^L\, dp_n{}^L$ times the longitudinal phase-space element for $n-2$ particles.

The two momenta $p_1{}^L$ and $p_n{}^L$ may now be generated with uniform distribution from zero to their maximum values. The direction of each is also chosen randomly. Then the remaining $n-2$ particle momenta are generated according to uniform longitudinal phase space, as described in the previous section. The total longitudinal weight then becomes

$$W^L = p_{1\max}^L p_{n\max}^L W_u{}^L, \tag{85}$$

where $W_u{}^L$ is the longitudinal weight for $n-2$ particles generated uniformly as given by Eq. (81). The two factors $p_{n\max}^L$ and $p_{n\max}^L$ are the maximum possible momenta for the first and nth produced particles. If $p_1{}^L$ is generated first, then the value of $p_{n\max}^L$ will depend on the value of $p_1{}^L$.

This weight factor will, of course, be a function of the total energy of the $n-2$ particle system of nonleading particles. This total energy will, in turn, depend on the generated momenta of the leading particles. Therefore, in principle, the distribution of generated events still contains some dependence on the leading particle momenta. In fact, however, the authors have found this dependence to be sufficiently weak that this generation method is suitable for any practical application.

The leading particles may, in fact, be given distributions which are any function of the Feynman x variable. If we want the distributions of $p_1{}^L$ and $p_n{}^L$ to be given by $f_1(x_1)$ and $f_n(x_n)$, respectively, we use a longitudinal distribution function given by

$$\begin{aligned} F_L &= E_1 E_n f_1(x_1) f_n(x_n) \\ &= E_1 E_n f_1(p_1{}^L/W) f_n(p_n{}^L/W), \end{aligned} \tag{86}$$

where W is the total center-of-mass energy.

The two leading particle momenta are again broken off as in Eq. (84), which now becomes

$$F_L\, d\rho_L = \frac{1}{4}\, W^2\delta^2\left(P^L - p_1{}^L - p_n{}^L - \sum_{i=2}^{n-1} p_i{}^L\right) f_1(x_1)\, f_n(x_n)\, dx_1\, dx_n \prod_{i=2}^{n-1} \frac{dp_i{}^L}{2E_i}\,. \tag{87}$$

The two x values are now determined in terms of two uniformly distributed random numbers r_1 and r_n by

$$\begin{aligned} \int_{-x_{1\max}}^{x_1} f_1(\zeta)\, d\zeta &= r_1 I_1\,, \\ \int_{-x_{n\max}}^{x_n} f_n(\zeta)\, d\zeta &= r_n I_n\,, \end{aligned} \tag{88}$$

where

$$I_1 = \int_{-x_{1\max}}^{x_{1\max}} f_1(\zeta)\, d\zeta,$$
$$I_n = \int_{-x_{n\max}}^{x_{n\max}} f_n(\zeta)\, d\zeta. \tag{89}$$

The limits of the integral of f_n depend on the generated value of x_1. They are essentially the extreme values of x_n kinematically possible given the generated value of x_1. The remaining $n - 2$ particles may then be generated according to uniform longitudinal phase space, as described in the previous section. The total weight for an event then becomes

$$W^{\mathrm{L}} = \tfrac{1}{4} W^2 I_1 I_n W_u{}^{\mathrm{L}}. \tag{90}$$

The leading particle distribution functions may be any of a wide variety of possibilities. If Eqs. (88) are solvable for x_1 and x_n, the leading particle momentum values may be generated analytically. Possible examples of such distributions might be $(1 - x)^{\alpha}$, e^{-ax}, or $a + bx$.

It should also be mentioned that it is possible to generate a leading particle momentum according to any finite sum of invertible distributions. We shall illustrate the method for the sum of two distributions. The case of the sum of any finite number of distributions then follows by induction.

Consider the case

$$f_1(x) = f_{1a}(x) + f_{1b}(x). \tag{91}$$

Then we have

$$I_1 = I_{1a} + I_{1b}, \tag{92}$$

where each I represents the integral of the similarly subscripted f function over the available kinematic interval. We now choose a random number r generated with uniform distribution from 0 to 1. If we have

$$rI_1 \leqslant I_{1a}, \tag{93}$$

we generate x_1 according to distribution f_{1a}. Otherwise we generate x_1 according to distribution f_{1b}.

Single leading particle distributions may also be generated in an obvious fashion. An essentially uniform distribution may be obtained by choosing the longitudinal weighting function equal to the center-of-mass energy of the leading particle. Then only one particle would be split off from the remainder in Eq. (84). It could also be given any desired x distribution. The remaining particles would then be generated according to uniform longitudinal phase space.

E. *Longitudinal Clustering*

As described in Section 4B, the ensemble of n particles may be decomposed into single particles by any possible tree. The phase-space factors associated with each generated intermediate mass are then given in Eq. (74) and the text following.

Some (or all) of these intermediate multiparticle states may actually represent physical clusterings of particles and therefore may be reflected in the longitudinal distribution function. In generating longitudinal momenta, therefore, we must take into account both the longitudinal distribution function and the phase-space factors.

It is important, therefore, to distinguish between the form of the distribution function and that of the final distribution. Because of the phase-space factors, a distribution function which is a given function of certain kinematic variables need not produce final distributions which are the same (or even similar) functions of the same kinematic variables. As seen in the section on leading particle behavior, a nonuniform distribution function is needed in order to generate essentially uniform leading particle longitudinal momentum distributions.

Because of these considerations, the method of generation may depend on how sharply structured the distribution function F_L is compared with the underlying phase-space factors. If the structure of the amplitude is very sharp (as is the case with a narrow resonance), it may be appropriate to generate the cluster masses according to the distribution function F_L alone. The phase-space factors are then evaluated and contribute to the final longitudinal weight W^L of the generated event. The variation of these factors over the kinematic range of generated events will then be small, and the generation efficiency will be satisfactory.

If the rate of variation of the function F_L and that of the underlying phase-space factors are comparable, it is necessary to take both into account in the event generation. It is best to illustrate the application of our method with a simple example.

We consider the formation of two longitudinal clusters as shown in Fig. 7. Each cluster might decay by the successive emission of single particles. We assume that individual particles are given off with an exponential momentum distribution with a scale factor of a. We then consider only the effect on the total mass distribution of each of the clusters.

It is then reasonable to assume that the factor for each cluster which enters into the distribution function F_L is given by

$$F = e^{-M/a(k-1)}, \tag{94}$$

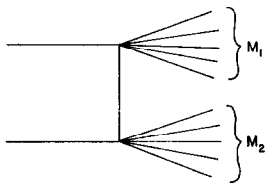

FIG. 7. The production of secondary particles in two longitudinal clusters. The invariant masses of these clusters are M_1 and M_2.

where M is the mass of the cluster and k is its multiplicity. For an ensemble of n particles which forms two clusters of k and $n - k$ particles, the total distribution function F_L is then given by

$$F_L = e^{-M_1/a(k-1)} e^{-M_2/a(n-k-1)}. \tag{95}$$

Therefore we wish to generate M_1 and M_2 by distributions which correspond as closely as possible to the product of the exponential times the phase-space factor. For the first cluster this distribution becomes

$$f_1 = e^{-M_1/a(k-1)} \tau_1^{(k-3)/2} (1 - \tau_1)^{(n-k)/2-1}, \tag{96}$$

while for the second we have

$$f_2 = e^{-M_2/a(n-k-1)} \tau_2^{(n-k-3)/2} (1 - \tau_2)^{-1/2}. \tag{97}$$

We let S_1 and S_2 be the sums of the longitudinal masses of the particles composing clusters 1 and 2, respectively. Then the relations between the masses M and the τ's is as follows.

$$\begin{aligned} M_1 &= (W - S_1 - S_2)\, \tau_1 + S_1 , \\ M_2 &= (W - M_1 - S_2)\, \tau_2 + S_2 , \end{aligned} \tag{98}$$

where W is the total center-of-mass energy of the system. We can remove the pole in τ_2 by a transformation of variables

$$\tau_2 = \sigma(2 - \sigma), \tag{99}$$

so that if we now generate in σ, the appropriate distribution becomes

$$f_2' = e^{-M_2/a(n-k-1)} \sigma^{(n-k-3)/2} (2 - \sigma)^{(n-k-3)/2}. \tag{100}$$

Since the masses are exponentially damped, at high energies the generated values of τ_1 and σ will almost always be small, and the factors of $1 - \tau_1$ and $2 - \sigma$ will suffer little variation. For the same reason we can generate τ_2 and σ from zero to infinity. If a mass value generated does not fall within the kinematic bounds, we discard that value and generate another. With the exponentially damped masses and at high energy, the probability that a generated mass will not fall within the kinematically allowed limits is vanishingly small.

We therefore generate two random numbers r_1 and r_2 according to the distribution

$$f''(r) = r^i e^{-r}, \tag{101}$$

where i_1 is the greatest integer equal to or less than $(k - 3)/2$, and i_2 bears the same relationship to $(n - k - 3)/2$. The generation may be done by taking [13]

$$r = -\log \prod_i s_i , \tag{102}$$

where $\prod s_i$ is the product of $i + 1$ random numbers, each generated with uniform distribution from zero to one. The variables τ_1 and σ may now be evaluated as

$$\tau_1 = \frac{a(k-1)}{(W - S_1 - S_2)} r_1 ,$$
$$\sigma = \frac{a(n-k-1)}{2(W - M_1 - S_2)} r_2 . \tag{103}$$

From these expressions the invariant masses may be evaluated using Eqs. (98) and (99).

We may then decompose the two clusters according to uniform longitudinal phase space. The final longitudinal weight W^{L} for the event involves the Jacobians of all variable transformations and all discarded terms. It therefore becomes

$$W^{\mathrm{L}} = \frac{1}{R_n} (E_0 - S_n)^{(n-3)/2} k! (n-k)! \left[\frac{a(k-1)}{(W - S_1 - S_2)}\right]^{j_1+1}$$
$$\times \left[\frac{a(n-k-1)}{2(W - M_1 - S_2)}\right]^{j_2+1} e^{-S_1/a(k-1)} e^{-S_2/a(n-k-1)} \tau_1^{j_1} (1-\tau_1)^{(n-k)/2-1}$$
$$\times \sigma^{j_2} (2-\sigma)^{(n-k-3)/2} e^{(W-M_1-S_2)\sigma^2/a(n-k-1)} I_k I_{n-k} . \tag{104}$$

The exponent j_1 will be 0 if $(k-3)/2$ is integer and $\frac{1}{2}$ otherwise, while the exponent j_2 bears the same relationship to $(n-k-3)/2$. The factor R_n is the term defined in Eq. (71) applied to the primary vertex. The two factors I_k and I_{n-k} represent the phase space factors for the decomposition of the two clusters. As shown in Eq. (81), I_k, for example, may be written as

$$I_k = \left[\prod_{i=1}^{k} \frac{2\mu_i}{R_i}\right] I_{wk} . \tag{105}$$

This method can easily be extended or modified to include other longitudinal structure. One might wish to generate events in which three clusters are formed as shown in Fig. 8. One would first generate the invariant mass M_1 of the first cluster

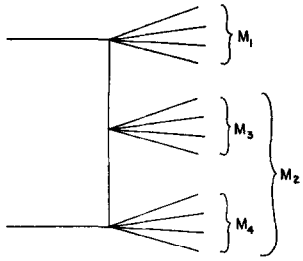

FIG. 8. The production of secondary particles in three longitudinal clusters. The invariant mass of the first cluster is M_1, while that of the second and third together is M_2. The individual invariant masses of the second and third clusters are M_3 and M_4, respectively.

and the invariant mass M_2 of the other two clusters taken together. Then one would decompose M_2 by generating the individual invariant masses M_3 and M_4 of the two remaining clusters. Finally, one could decompose each of the three clusters into single-particle states, using either uniform longitudinal phase space or some other desired mechanism.

5. Resonances

The method of production of particles with limited transverse momentum can be extended to include resonances. The amplitude limiting the transverse momentum of secondaries would then apply to the momentum of the resonances. The resonances could then be made to decay according to any desired distribution in their own center of mass.

Therefore the first step in event generation would be to determine the mass of each resonance. The phase-space distribution function for a resonance mass is given in terms of a Breit–Wigner function

$$\mathrm{BW}(M_\mathrm{R}) = \frac{1}{[(M_\mathrm{R} - \mu_\mathrm{R})\, 2/\Gamma_\mathrm{R}]^2 + 1}. \tag{106}$$

The quantity μ_R gives the central mass of the resonance, while Γ_R gives its width. The combined distribution function and differential phase-space element then become

$$F\, d^n\rho = \left[\prod_i \mathrm{BW}(M_{\mathrm{R}_i})\, D_i(p_{\mathrm{R}_i})\right] F_\mathrm{L}(\{p^\mathrm{L}\}, \{p^\mathrm{T}\})\, F_\mathrm{T}(\{p^\mathrm{T}\})\, d^n\rho. \tag{107}$$

The factor $D_i(p_\mathrm{R.})$ gives the distribution function for the decay of the ith resonance. The differential phase-space element may now be factored to isolate the mass of a resonance

$$d^n\rho = dM_\mathrm{R}{}^2 d^k\rho_\mathrm{R} d^{n-k+1}\rho. \tag{108}$$

The factor $d^k\rho_\mathrm{R}$ is the phase space associated with the decay of a resonance of k particles. This procedure may be iterated until the masses of all resonances are isolated:

$$d^n\rho = d^N\rho \prod_i dM^2_{\mathrm{R}_i}\, d^{k_i}\rho_{\mathrm{R}_i}. \tag{109}$$

The number N gives the sum of the number of primary resonances plus the number of directly produced stable particles. The momenta in $d^N\rho$ are then just those occurring in the distribution functions F_L and F_T, described in Section 2.

To generate the mass of the resonances we use a slight modification of a well-known method [5]. The mass is given in terms of a random number r, uniformly distributed from zero to one, as

$$\frac{\Gamma}{2}\left[\tan^{-1}\left(\frac{M_\mathrm{R} - \mu_\mathrm{R}}{\Gamma/2}\right) - \tan^{-1}\left(\frac{S - \mu_\mathrm{R}}{\Gamma/2}\right)\right] = rH, \tag{110}$$

where H is the maximum value of the expression on the left:

$$H = \frac{\Gamma}{2}\left[\tan^{-1}\left(\frac{M_{\max} - \mu_R}{\Gamma/2}\right) - \tan^{-1}\left(\frac{S - \mu_R}{\Gamma/2}\right)\right]. \tag{111}$$

The quantity S is the sum of the rest masses of the stable particles which comprise the resonance. The quantity $M_{\max}$ is the maximum kinematically allowed mass for the resonance.

The masses of the resonances produced in the reaction may be successively generated by the above method. The quantity H for each resonance will be a factor in the total weight of the event. One may then decompose the phase space of the resonance using techniques described by Friedman [5].

At high energies the maximum kinematically allowed mass $M_{\max}$ of the resonance can be quite large. As the mass of a resonance is increased beyond several full widths above its central value, the weight for the event begins to increase, owing to the phase-space factors in $d^k\rho_R$. A histogram of generated masses will then show a high-energy rise, well beyond the normal peak of the resonance at its central value.

This behavior is not ordinarily thought of as part of the resonant structure. Therefore, we have simply given the mass of the resonance a high-energy cutoff of three full widths above its central value. The quantity $M_{\max}$ is then given by

$$M_{\max} = \mu + 3\Gamma, \tag{112}$$

if this value is smaller than the kinematically allowed maximum.

Given the resonance masses, one can then generate the four-momenta of all the resonances and directly produced stable particles. The resonances can be made to decay by means of lower-energy phase-space techniques. Finally, the weight for each event will be just the total weight for the peripheral event generation times a product of weighting factors for the resonances. The weight factor for each resonance will consist of the product of its H factor, as shown in Eq. (111), and the weight from its phase space decomposition.

6. Concluding Remarks

The methods described in the preceding sections represent both general procedures and specific techniques for generation according to certain distribution functions. It is clear that not everything described can be written into a single computer program.

However, all specific distributions mentioned above have been programmed by the authors and used to generate events. Many have been incorporated into the Monte Carlo phase-space computer program NVERTX [18]. Others have been programmed by the temporary replacement of an NVERTX subroutine. Essentially any distribution which can be generated by the methods described in this paper can be programmed by the latter method. For specific details the reader should consult the NVERTX manual [19].

For Gaussian damped transverse momentum the generation subroutines employ the method of Jadach [10]. For continuity with low energies, he reduces the magnitude of the transverse momentum scale factor as the energy is lowered. However, as used by the present authors, the final weight function represents generation by means of the transverse scale factor specified by the user.

The generating subroutines are also sufficiently self-contained that they can be used as a separate package. The authors have also made use of them in this mode. Their use is simple enough that little additional programming is required.

This latter mode of event generation also allows the user to vary the multiplicity. A specified number of events of a given multiplicity may be generated and the desired distributions obtained. These distributions may then be renormalized by use of the average weight of the generated sample. Finally, the distributions may be combined with those from other multiplicities. The high efficiency of the generation methods described here allow statistically accurate determination of the renormalization factor. The authors have generated events in this manner also.

Although we have described a variety of methods, we have limited ourselves to those distributions where the longitudinal and transverse momentum components are in some sense separable, and the transverse components are damped. For these cases, the methods described give a high degree of generation efficiency. For other distributions, techniques such as others in NVERTX or the Monte Carlo phase-space program FOWL [20] may be appropriate.

REFERENCES

1. F. E. TAYLOR *et al.*, *Phys. Rev.* **14** (1976), 1217.
2. F. JAMES, CERN Yellow Report No. 68-15 (1968); E. BYCKLING AND K. KAJANTIE, "Particle Kinematics," Wiley, New York, 1973.
3. G. KOPYLOV, *Z. Eksper. Teoret. Fiz.* **39** (1960), 1091, translation *Soviet Physics JETP* **12** (1961), 761.
4. D. T. GILLESPIE, Doctoral Dissertation, Appendix (1968).
5. J. H. FRIEDMAN, *J. Computational Phys.* **7** (1971), 201.
6. E. BYCKLING, M. KAARTINEN, K. KAJANTIE, AND H. VILLANEN, *J. Computational Phys.* **4** (1969), 521.
7. J. H. FRIEDMAN, G. R. LYNCH, C. G. RISK, AND T. G. ZANG, JR., *J. Computational Phys.* **8** (1971), 144.
8. O. PENE AND A. KRZYWICKI, *Nucl. Phys. B* **12** (1969), 415.
9. W. KITTEL, L. VAN HOVE, AND W. WOJCIK, *Comput. Phys. Commun.* **1** (1970), 425.
10. S. JADACH, *Comput. Phys. Commun.* **9** (1975), 297.
11. R. P. FEYNMAN, *Phys. Rev. Lett.* **23** (1969), 1415.
12. M-S. CHEN AND R. F. PEIERLS, *J. Computational Phys.* **16** (1974), 195.
13. C. J. EVERETT AND E. D. CASHWELL, Los Alamos Report LA-5061-MS (1972).
14. F. W. BUSSER *et al.*, *Phys. Lett. B* **51** (1974), 306; M. D. NEGRA *et al.*, *Phys. Lett. B* **59** (1975), 401; K. EGGERT *et al.*, *Nucl. Phys. B* **98** (1975), 73; G. FINOCCHIARO *et al.*, *Phys. Lett. B* **50** (1974), 396.
15. D. SIVERS, S. J. BRODSKY, AND R. BLANKENBECLER, *Phys. Rep.* **23** (1976), 1.
16. R. HAGEDORN, "Relatistic Kinematics," Benjamin, New York, 1964.
17. M. G. ALBROW *et al.*, *Nucl. Phys. B* **54** (1973), 6; P. CAPILUPPI *et al.*, *Nucl. Phys. B* **79** (1974), 189.

18. C. A. BORDNER, JR., A. E. BRENNER, AND E. E. RONAT, *Rev. Sci. Instrum.* **37** (1966), 36.
19. A. E. BRENNER, D. C. CAREY, R. PORDES, AND J. H. FRIEDMAN, Fermilab Computer Dept. Report PM-4.
20. F. JAMES, FOWL, CERN Computer Centre Program Library Long Write-Up W505.